TWO BIDDER PROCUREMENT AUCTION WITH ASYMMETRIC POST-AUCTION BANKRUPTCY RISK

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The possibility of default in procurement auctions is a rampant economic issue. Although auction design in the presence of post-auction bankruptcy risk has been investigated in the past literature, it was commonly assumed that bidders are ex-ante identical. This paper investigates a special case of two-bidder procurement with asymmetric post-auction bankruptcy risk, yielding an asymmetric equilibrium. Two standard auctions have been compared; the first- and second-price auction. I confirm that if the risk and cost of bankruptcy are high, the first-price auction generates higher revenue given the procurer highly values the project. I also show that as opposed to the past literature, the first-price auction may induce a cheaper procurement price. Moreover, the bidder exposed to default risk - if his type is advantageous - is quite confident of winning that he may bid less aggressive than his counterpart.

KEYWORDS: Asymmetric auction, post-auction bankruptcy, procurement auction.

1. INTRODUCTION

THE POSSIBILITY OF DEFAULT in procurement auctions is a rampant economic issue. During 1990-1997 in the US, bankrupt contractors left both private and public projects unfinished with liabilities more than $21 billions[2)]. While auction mechanisms are increasingly acknowledged as an imperative tool for efficient or lucrative allocations of large resource for both government and firms, welfare costs attributed to unexpected bankruptcies have often been neglected by academic studies. Traditional auction theories postulate that bidders are affluent: bidders in art auctions, for example, often involved high-income individuals, investors and brokers. Here, competition among bidders basically yields more aggressive bidding behavior which brought higher revenues for the auctioneers; however, it is relatively a recent understanding that bidders with limited liabilities bid more aggressively *because* they have the option to file bankruptcy and eliminate downside losses. This poses a threat both to the auctioneer and the economy.

What has also been overlooked is the possibility of asymmetric bidders. In the conventional assumption of symmetry, bidders' "types" - or their private

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[2)] Dun & Bradstreet Business Failure Record valuations of a good - are drawn from an independent and identical distribution so that in each bidder's and auctioneer's point of view, they are ex-ante identical. This assumption ensures the existence of a symmetric equilibrium (provided other conditions are sufficiently met). In reality, this is nary the case: for example, in major art auctions and contract bidding, bidders obviously have ex-ante heterogeneous budget constraints and preferences yielding asymmetric beliefs about each other. Asymmetric auctions are far from being thoroughly explored since there is no general form of asymmetry and even a very minor introduction of heterogeneity may engender anomalous behavior. Thus, a general closed form of equilibrium conditions are hard to be found.

This paper investigates a special case of two-bidder procurement with asymmetric post-auction bankruptcy risk, yielding an asymmetric equilibrium. The contribution of this paper is in integrating the problem of post-auction bankruptcy risk with that of asymmetry. Past literature on post-auction default risk has lack of understanding of bidding strategies in the presence of asymmetry since the assumption that bidders are ex-ante identical - yielding a symmetric equilibrium - has been commonly used. Furthermore, the formulation of the bankruptcy cost is limited in the sense that the auctioneer bears no cost at all, that the cost has only been focused on recovery rate of the winning payment, or that the cost is summarized by a constant. I formulate two cost parameters which determine the payment recovery rate - the payment transferred from the winning bidder back to the auctioneer at the advent of default and the procurement rate at which procurer partially receives the good, respectively.

Two standard auction mechanisms have been compared; the first-price and second-price auction. I confirm that if the risk and cost of bankruptcy are high, the first-price auction generates higher revenue. I also show that as opposed to the past literature, the first-price auction induces a cheaper procurement price when bankruptcy risk is large. Moreover, when the risk is small, the bidder exposed to post-auction risk can bid less aggressive than his counterpart.

The outline of this paper is as follows. Section 2 presents a brief review on the related literature. In Section 3, I present the model. Section 4 contains the equilibrium conditions of the FPA and SPA, and a special case where the corresponding bidding strategies are explicitly analyzed. Section 5 provides the auctioneer's revenue comparisons using the equilibrium of the special case in the previous section. Section 6 concludes. All proofs are in Appendix.

2. RELATED LITERATURE

Growing literature ponders upon auction designs in the presence of post-auction bankruptcy risk. One of the main contributors is Board (2007) who presents auctions in the presence of symmetric bankruptcy risk discusses the bidding behavior under the high-bid FPA and SPA. In his model, bidders are ex-ante identical: bidders' valuations upon an item owned by a seller are independently and identically subject to an identical exogenous shock which is to be realized after the auction winner is announced. He shows that when bidders have limited liability, higher price and higher probability of bankruptcy are induced by the SPA.

Burguet et al. (2009) investigate mechanism designs in procurement auctions. They show that every feasible mechanism has a higher chance of selecting the financially weakest bidder. Although they introduce heterogeneous initial wealth of the bidders, it is drawn from an independent and identical distribution. Thus, if this wealth is interpreted as bidders' type, it is a procurement auction under ex-ante identical bidders and identical default risk.

Furthermore, while many previous studies focus on costless bankruptcy for the procurer, Board (2007) relaxes the assumption with post-bankruptcy recovery rate and Burguet et al. (2009) introduces a constant bankruptcy cost; both papers show that the procurer's choice of auction mechanism critically depends on bankruptcy cost. However, Board's (2007) recovery rate is the proportion of the winning payment[3] that is transferred to the auctioneer when the winner declares bankruptcy. The value of the good, on the other hand, is kept safe by the auctioneer. This may be a reasonable assumption since in high-bid auctions, auctioneers have the good at hand ex-ante. In procurement auctions, the auctioneer does not possess the good, but opens the auction to allocate his resources and delegate production or a project to the winning bidder. In Burguet et al. (2009), the auctioneer always procures the good and retrieves all the winning payment regardless of the declaration of default. In the next section, I introduce two separate cost parameters that determines the payment recovery rate and the procurement rate, respectively - which becomes a crucial variable determining the auctioneer's choice of mechanism.

The models in Board (2007) and Burguet et al. (2009) both yield a symmetric equilibrium. Maskin and Riley (2000) present an extensive analysis on asymmetric auctions (in the absence of bankruptcy risk) where asymmetric equilibriums are derived from various forms of bidder-asymmetry. The main difference between their model and the one in this paper is the timing of the asymmetry realization: while their asymmetry is realized before the auction - determining bidders' type, the model to be presented below realizes an asymmetric post-auction shock that updates bidders' type.

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[3] Board (2007) interprets this payment as the bankrupt asset.

3. MODEL

Consider a two-bidder procurement auction. Before the auction, each risk neutral bidder i observes a private cost signal $c_i \in [\underline{c}, \overline{c}]$ with $\underline{c} > 0$ which are distributed independently and identically with distribution function F and density f. After the auction winner is announced, bidder 2, in addition, faces a privately observed post-auction exogenous shock $s \in [-z, z]$, $(z > 0)$ which is distributed independently of c_i's with distribution function $G(s)$ and density $g(s)$ so that his cost valuation is $c_2 + s$. If bidder 2 wins and if $c_2 + s$ is smaller than the winning payment, he will produce the item for the procurer. If $c_2 + s$ is larger, he declares bankruptcy and earns nothing[4]. Bidders have limited liability; without loss of generality, I assume that bidders possess no initial financial asset so that the loss of bidder 2 entering the auction is at most zero. Payoffs of bidder 1 and 2 conditional on winning are $P - c_1$ and $\max\{P - c_2 - s, 0\}$, respectively, where P is the winning payment.

The procurer's maximum willingness to pay (WTP) for the item is given as a positive number v_0. She can choose her auction mechanism among two choices - the FPA and SPA. Past literature does not provide a commonly implemented setting of the procurer's expected revenue in the presence of post-auction bankruptcy[5]. In this model, if bankruptcy occurs, her revenue depends on an exogenous recovery rate r and an exogenous completion rate d. The former is the portion of the payment to the winner that is reclaimed by the procurer in the presence of bankruptcy. This may represent bankruptcy-specific costs such as litigation costs, liquidation costs[6], etc. The latter is the portion of production completed at the time of the bankruptcy declaration. This may represent production-specific costs such as delays in the completion of the item, cost of new procurement process, etc. Thus, the expected revenue of the procurer is

$$\begin{aligned} R(m; d, r, v_0) &= [1 - P_B(m)](v_0 - P(m)) + P_B(m)(dv_0 - (1-r)P(m)) \\ &= v_0(1 - (1-d)P_B(m)) - P(m)(1 - rP_B(m)) \end{aligned}$$

where $P(m)$ and $P_B(m)$ are the expected payment and the probability of bankruptcy, respectively, given a mechanism $m \in \{SPA, FPA\}$.

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4) We can consider bidder 1 as a large construction firm that is less vulnerable to economic shocks represented by s in the model while bidder 2 is a mid or small-sized firm, of which cost valuation more vulnerable to macroeconomic conditions.

5) While many previous studies focus on costless bankruptcy for the procurer, Board (2007) relaxes the assumption with post-bankruptcy recovery rate and Burguet et al. (2009) introduces a constant bankruptcy cost; both papers show that the procurer's choice of auction mechanism critically depends on bankruptcy cost.

6) White (1989) reports direct administrative costs for liquidation of a bankrupt firm in the US account for 7.5%-21% of the liquidation proceeds.

The following is the timing of the model summarized:

1) Nature chooses the cost valuation c_i privately observed by bidder i.
2) The procurer announces the procurement mechanism.
3) Bidders submit their bids, and the payment P is given to the winner who is chosen to be the producer of the item.
4) If bidder 1 is the winner, the procurer obtains the item. If bidder 2 wins, however, the exogenous shock s is realized. If the shock s is such that $c_2+s \leq P$, then bidder 2 finishes the item. Otherwise, he declares bankruptcy.
5) The procurer and the winner realize their payoffs.

4. EQUILIBRIUM BIDDING STRATEGIES

4.1 *The Second-Price Auction*

I analyze the Bayesian Nash equilibrium (BNE) bidding strategies of two standard auction mechanisms; the FPA and SPA. The ones for the SPA are straightforward as follows.

PROPOSITION 1: *Consider a two-bidder SPA. Assume independently distributed $c_i \sim F, f$ over $[\underline{c}, \bar{c}]$ and $s \sim G, g$ over $[-z, z]$ and $z < \underline{c}$. Then, the BNE bidding strategies of bidder 1 and 2 are*

$$B_1(c_1)=c_1 \ \textit{and} \ B_2(c_2)=c_2-z \quad \forall c_1, c_2 \in [\underline{c}, \bar{c}]$$

PROOF: See Appendix

Proposition 1 presents the BNE bidding strategies of bidder 1 and 2 under the SPA, which are analogous to the one in Board (2007), where bidders with zero initial wealth in the SPA submit their bid as if they are expecting the luckiest scenario the shock s can realize, *i.e.* when $s=-z$. It is also congruent with the past literature where bidders with the option of declaring bankruptcy bid more aggressively while bidder 1's strategy is equivalent to the one in the SPA with no post-shock case.

4.2 *The First-Price Auction*

Suppose $b_1(c_1)$ and $b_2(c_2)$ are the BNE bidding strategies of bidder 1 and 2 in the FPA. Further suppose that these are increasing, differentiable and have inverses $\phi_1 \equiv b_1^{-1}$ and $\phi_2 \equiv b_2^{-1}$, respectively. The expected utility of bidder $i=1,2$

with cost valuation c_i and bidding amount b_i is

$$u_1(b_1;c_1) = (b_1 - c_1)[1 - F(\phi_2(b_1))] \text{ and}$$
$$u_2(b_2;c_2) = E_s(\max\{b_2 - c_2 - s, 0\})[1 - F(\phi_1(b_2))],$$

respectively. Then for each c_1 and c_2, $b_1(c_1)$ and $b_2(c_2)$ solve $\max_{b_1} u_1(b_1;c_1)$ and $\max_{b_2} u_2(b_2;c_2)$. Then, bidder 1's first order condition is

$$1 - F(\phi_2(b_1)) - (b_1 - c_1) f(\phi_2(b_1))\phi_2'(b_1) = 0 \text{ at } b_1 = b_1(c_1)$$

However, bidder 2's condition $du_2(b_2;c_2)/db_2 = 0$ at $b_2 = b_2(c_2)$ is of twofold:

$$1 - F(\phi_1(b_2)) = \left(b_2 - c_2 - z + \int_{-z}^{z} G(s)ds\right) f(\phi_1(b_2))\phi_1'(b_2) \text{ at } b_2 = b_2(c_2) \geq c_2 + z$$

and

$$G(b_2 - c_2)(1 - F(\phi_1(b_2))) = \left(\int_{-z}^{b_2 - c_2} G(s)ds\right) f(\phi_1(b_2))\phi_1'(b_2) \text{ at } b_2 = b_2(c_2) < c_2 + z.$$

The first term of bidder 2's second condition is the marginal benefit of a higher bid due to decreasing probability of bankruptcy, while in the first condition, bankruptcy risk is zero. The second term is the marginal cost due to decreasing probability of winning.

Due to the twofold condition of bidder 2's, when constructing the BNE strategies for the FPA, I find the following definition useful.

DEFINITION 1: A subset I_B of $[\underline{c}, \bar{c}]$ is called a *set of risky values* if $b_2(c_2) < c_2 + z$ for all $c_2 \in I_B$.

When $c_2 \in I_B$, bidder 2's bankruptcy risk is greater than zero. By the boundary condition and the increasing property of $b_2(\bullet)$, it is trivial that I_B is nonempty whenever $z > 0$.

Without constructing any further assumptions on distribution F and G, explicit functional solutions to the above system of differential equations cannot generally be obtained. However, we can derive a boundary condition of

$$\bar{c} - z < b_2(\bar{c}) < b_1(\bar{c}) \leq \bar{c} \text{ [7].}$$

[7] See Appendix for proof.

and the following single-cross property.

PROPOSITION 2: *Assume* $c_i \sim F, f$ *over* $\left[\underline{c}, \bar{c}\right]$ *and* $s \sim G, g$ *over* $[-z, z]$ *with* $E(s) = 0$. *Further assume* z *is large enough so that* $I_B = [\underline{c}, \bar{c}]$ *holds. Then,*
(1) $\phi_1(b)$ *and* $\phi_2(b)$ *intersect at most once.*
(2) And if they do not intersect, $b_1(c) > b_2(c)$ *for all* $c \in \left[\underline{c}, \bar{c}\right]$

PROOF: See Appendix

In an asymmetric auction where a strong bidder 1's values are stochastically advantageous[8] than those of bidder 2, the "weak" bidder 2 bids more aggressively than the "strong" bidder 1, globally. However, Proposition 2 suggests that in the presence of post-auction bankruptcy risk - while the value distributions of the two bidders are identical - bidder 2 may bid less aggressively than bidder 1. If we assume the post-shock $s \sim U[-z, z]$, we can derive the following corollary.

COROLLARY 1: *Assume* $c_i \sim F, f$ *over* $\left[\underline{c}, \bar{c}\right]$ *and* $s \sim U[-z, z]$ *with* $0 < z \leq \underline{c}$ *and* $z \leq \bar{c} - \underline{c}$. *Further assume* z *is large enough so that so that* $I_B = [\underline{c}, \bar{c}]$ *holds. Then,* $\phi_1(b)$ *and* $\phi_2(b)$ *intersect at most once. If they do not intersect,* $b_1(c) > b_2(c)$ *for all* $c \in \left[\underline{c}, \bar{c}\right]$ *and in addition, the distribution of bidder 2's bids dominate that of bidder 1's in terms of the hazard rate.*

PROOF: See Appendix

4.3 *Special Case: The FPA Under Uniform Distribution*

Imposing uniform distributions on distribution functions F and G can help solve the differential equations in (3.2) and also derive revenue implications for the procurer. I assume bidders' cost valuations $c_i \sim U[\underline{c}, \bar{c}]$ and the post-shock $s \sim U[-z, z]$ with $z < \underline{c}$[9]. To examine the effect of asymmetric post-auction shock, let us first consider a model with ex-ante identical bidders with and without bankruptcy risk. As shown in figure 1, the two symmetric bid functions in the presence and absence of bankruptcy risk is exceedingly different, the former being more aggressive. By intuition, the bid functions in the asymmetric post-shock case will be in-between the two.

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8) In a general model of asymmetric auction, it is assumed that the value distribution of a strong bidder dominates that of the weak bidder in terms of the hazard rate.

9) Without loss of generality, I further assume that $z \leq \bar{c} - \underline{c}$.

PROPOSITION 3: *Consider a FPA. Assume* $c_i \sim U[\underline{c}, \overline{c}]$ *and* $s \sim U[-z, z]$. *Further assume that* z *is large enough so that* $I_B = [\underline{c}, \overline{c}]$ *holds. Then,*

$$b_1(c_1) = \frac{c_1 + \overline{c}}{2} - \frac{z}{4} \text{ and } b_2(c_2) = \frac{c_2 + 2\overline{c}}{3} - \frac{z}{2} \text{ for } \forall z \in \left[\frac{4(\overline{c} - \underline{c})}{9}, \overline{c} - \underline{c}\right]$$

PROOF: See Appendix

Proposition 3 presents the asymmetric BNE bidding strategies in the FPA under the condition that the post-shock parameter z is large enough so that bidder 2 does not bid to a point where bankruptcy risk is zero. Notice that the asymmetry is at the different slopes and the right endpoints of the bid functions with respect to cost valuations c_i. (See figure 1) Bidder 1's highest bid always exceeds bidder 2's. This gap expands in z: both curves shifts down, but bidder 2's shift is faster as z rises. The slope of bidder 1's bid function is globally steeper: his marginal increase in bid with respect to a unit increase in cost valuation is higher than bidder 2's.

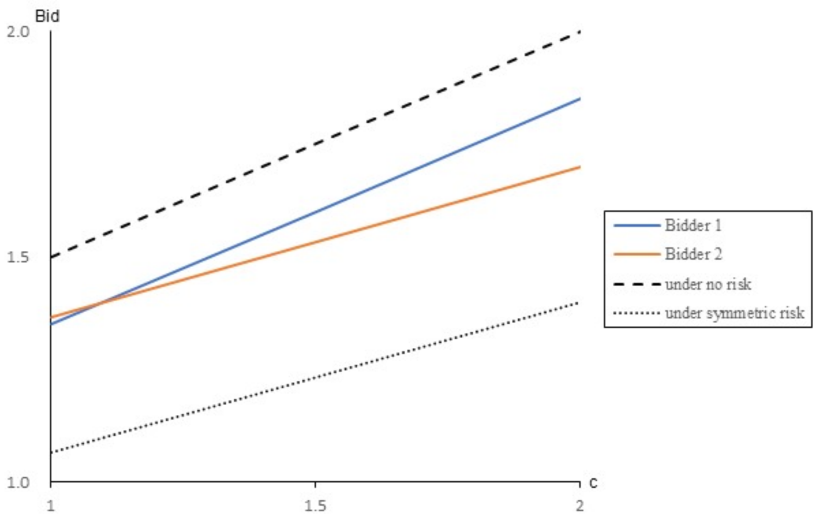


FIGURE 1. The BNE Bidding Strategies in FPA with Large Risk ($z = 0.6$)

The BNE bidding strategies of bidder 1(blue) and bidder 2(orange) in the FPA under asymmetric post-auction bankruptcy risk with $c_i \sim U[1, 2]$, $i = 1, 2$ and $s \sim U[-0.6, 0.6]$. The dashed line is the symmetric BNE bid function in the FPA under no post-shock, which is $b_{sym}(c) = (c + \overline{c})/2$ while the dotted line is the one under symmetric post-shock $s \sim U[-0.6, 0.6]$, which is $b(c) = (c + 2\overline{c})/3 - z$.

Thus, we can see the post-shock parameter z asymmetrically triggers a level effect: it spurs more aggressiveness from bidder 2 than from bidder 1, regardless of their cost-type. On the other hand, the slopes are not affected by z, suggesting that they depend more on distribution functions of c_i and s.

The aggressiveness of bidder 2 is mainly driven from the possibility of favorable post-auction shock. If, however, a favorable shock is not possible, he will bid less aggressively than he would in the absence of bankruptcy risk. Let's assume $s \sim U[-x, z]$ instead. Then, z in the bid functions is replaced by x[10], suggesting that the BNE strategies are functions of the magnitude of the highest favorable shock possible.

PROPOSITION 4: *Consider a FPA with $c_i \sim U[\underline{c}, \bar{c}]$ and $s \sim U[-z, z]$. Assume that z is small enough so that $I_B = [\underline{c}, \bar{c}]$ does not hold. Then, for $z < 4(\bar{c} - \underline{c})/9$,*

$$b_1(c_1) = \frac{(c_1 + \bar{c})}{2} - \frac{\hat{a}_2}{2} z \text{ and } b_2(c_2) = \begin{cases} \dfrac{(c_2 + \bar{c})}{2} - \dfrac{\hat{a}_2}{4} z & \text{if} \quad c_2 < c_0 \\ \dfrac{(c_2 + 2\bar{c})}{3} - \dfrac{(1 + \hat{a}_2)}{3} z & \text{if} \quad c_2 \geq c_0 \end{cases}$$

where $c_0 = \bar{c} - \frac{z}{2}\hat{a}_2 - 2z$ *and* $\hat{a}_2 = \left[(3(\underline{c} - \bar{c}) - z) + \sqrt{(3(\bar{c} - \underline{c}) + z)^2 + 8z^2}\right]/z.$

In particular, (1) $b_2(c_0) = c_0 + z$, (2) if $z \to 4(\bar{c} - \underline{c})/9$, then $\hat{a}_2 \to 1/2$ with which the BNE strategies are equivalent to those in Proposition 3, and (3) if $z \to 0$, then $\hat{a}_2 \to 0$ with which the BNE strategies are equivalent to those in the FPA with no bankruptcy risk: $b(c_i) = (c_i + \bar{c})/2 \quad \forall c_i$.

PROOF: See Appendix

Proposition 4 presents the asymmetric BNE bidding strategies under the condition that the post-shock z is small enough so that the bankruptcy risk is zero at some $c_2 \in [\underline{c}, \bar{c}]$. Now, with c_0 being the threshold, the slope of bidder 1's bid function is only locally steeper. (See figure 2) Notice that bidder 2's bid function is of twofold: at $c_2 < c_0$, bankruptcy risk is zero and the slope is equivalent to bidder 1's and at $c_2 \geq c_0$, bankruptcy is a possibility and the slope becomes the flatter one in Proposition 3.

The level effect of z is not equivalent to the ones in Proposition 3: at $c_2 \geq c_0$, bidder 2's curve shifts down faster than bidder 1's while at $c_2 < c_0$, bidder 1's shift is faster as z rises. Thus, when the risk is small, it triggers more aggressiveness from bidder 2 with high-cost type, and less from those with low - or advantageous - type.

Intuitively, when risk is quite small, a high-cost type bidder 2 concerns more about high probability of losing than bankruptcy risk and thus, becomes more aggressive. On the other hand, a low-cost type has a higher win probability and thus, concerns more about default risk and bid more conservatively. Recall that

10) That is, $b_1(c_1) = (c_1 + \bar{c})/2 - x/4$ and $b_2(c_2) = (c_2 + 2\bar{c})/3 - x/2$.

in the case of Proposition 3 - large risk case - even though the risk is large, by cutting off the downside loss, bidder 2 becomes more risk-loving and thus, his bid curve shifts down faster as z rises than bidder 1's regardless of their cost-type. This result - that there are circumstances where the bidder with post-auction bankruptcy risk becomes less aggressive than those without - is in opposition to past studies that assumed ex-ante identical bidders.

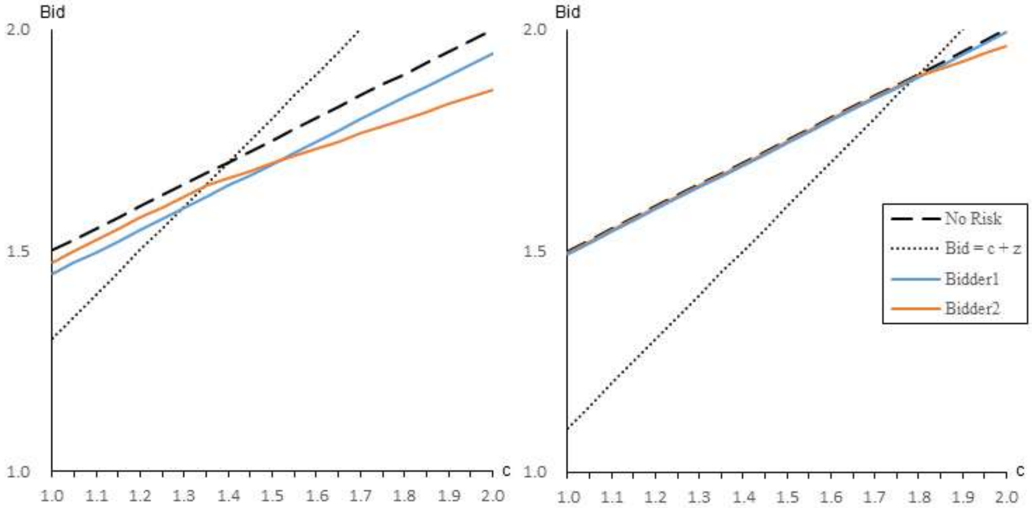


FIGURE 2. The BNE Bidding Strategies in FPA with Small Risk ($z=0.3$ and $z=0.1$)

The BNE bidding strategies of bidder 1(blue) and 2(orange) in the FPA with $c_i \sim U[1,2]$, $i=1,2$ when $s \sim U[-0.3, 0.3]$ (left) and $s \sim U[-0.1, 0.1]$ (right), respectively. The dashed line is the symmetric BNE bid function in the FPA under no post-shock. The dotted line is the curve $c+z$ which intersects the bidder 2's bid curve of c_0.

5. REVENUE COMPARISON: THE FPA VS. SPA

Now, consider the procurer's point of view. As mentioned above, the procurer's expected revenue depends on the exogenous recovery rate r and completion rate d. When $(d,r)=(1,1)$, bankruptcy is costless as many studies previously assumed: the procurer receives the final good and reclaims the payment for certainty. However, this is unlikely in the real world. Thanks to Board (2007) and Burguet et al. (2009), we acknowledge that bankruptcy cost has a critical impact on the auctioneer's attitude toward risk when designing an auction, and thus, revenue implications differ accordingly.

Under the uniform distribution assumption, we know that the probability of bankruptcy and the expected payment depends not only on the choice of auction mechanism, but also on the post-shock parameter z. Thus, the expected revenue of the procurer can be expressed as

$$R(m;d,r,v_0)=v_0[1-(1-d)P_B(m,z)]-P(m,z)\,[1-rP_B(m,z)]$$

where v_0 is the procurer's WTP for the item and $P(m,z)$ and $P_B(m,z)$ are the expected payment to the winner and the probability of bankruptcy, respectively, given a mechanism $m \in \{SPA, FPA\}$ and z.

As shown in figure 3, the expected winning payment $P(m,z)$ is decreasing in risk. When risk is small, the payment in the FPA is higher than the one in the SPA while when risk is large, it is the otherwise - as opposed to the result in Board(2007) that the SPA induces a higher price in a high-bidding auctions. This difference stems from the asymmetric setting of this model that induces a lower bound of the winning payment in the SPA. Even though bidder 2 bids more aggressively as z rises, the second lowest bid cannot be lower than bidder 1's true-telling bid. On the other hand, in the FPA, the winning payment decreases as bidder 2 becomes more aggressive.

The probability of bankruptcy is always higher in the SPA than the one in the FPA. This is because bidder 2 is always more aggressive in the SPA than in the FPA. However, the gap narrows as the level of risk rises. This implies that when the default risk is large enough, the difference in expected payments may play a more crucial role in procurer's choice of mechanism. However, this again depends on recovery and completion rate.

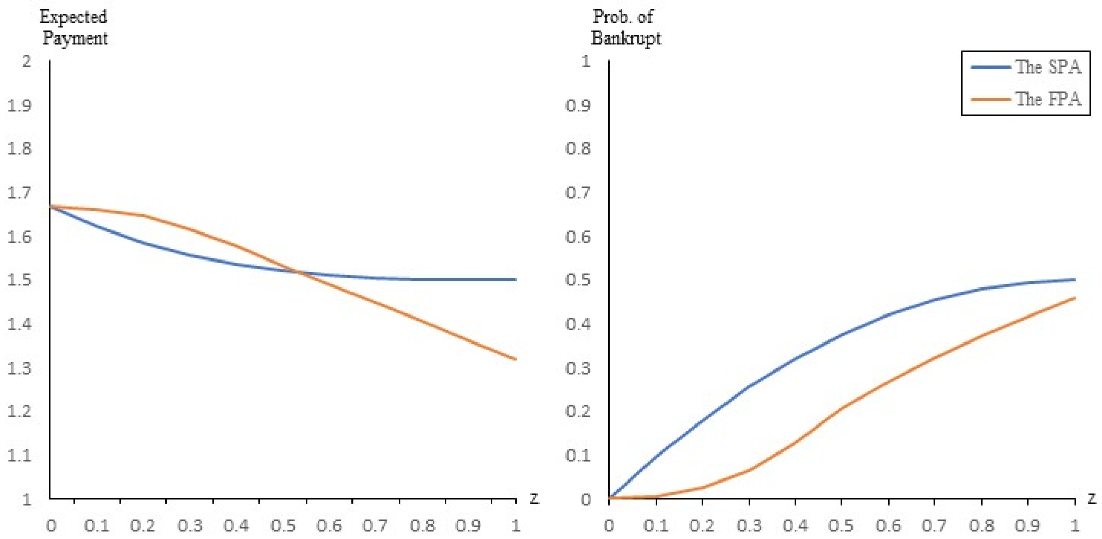


FIGURE 3. The Expected Winning Payment and The Probability of Bankruptcy in the FPA and SPA
The expected winning payment $P(m,z)$ (left) and the probability of bankruptcy $P_B(m,z)$ (right) in the FPA(orange) and SPA(blue) when $c_i \sim U[1,2]$, $i=1,2$

When the completion rate is $d=1$, the expected revenue is increasing in risk for both the FPA and SPA. This is consistent with previous studies that when bankruptcy is costless, the procurer becomes risk-loving. Furthermore, the FPA yields higher expected revenue for all v_0 when the bankruptcy risk is large enough *and/or* recovery rate r is small enough.

Notice the expected payment in the FPA becomes smaller than the one in the SPA as the risk parameter z grows while the difference in the probability of bankruptcy in both auctions seem to converge. The computation result shows when the completion rate is $d < 1$, there exists a revenue equivalent WTP $v_0{}^* = v_0{}^*(r, z)$ such that

$$R(FPA; d, r, v_0) \geq R(SPA; d, r, v_0) \text{ for all } v_0 \geq v_0{}^*$$

where equality holds only if $v_0 = v_0{}^*$. Furthermore, $v_0{}^*(r, z)$ is decreasing z and increasing in r. It implies that in the world of costly bankruptcy, despite the asymmetric setting of the model, there is a critical value $v_0{}^*$ of the procurer's WTP for the item that equalizes the revenues for the two mechanisms. And, the more she values the item, the more likely that she would prefer the FPA over the SPA. In another aspect, given her WTP fixed, if the risk and cost of bankruptcy are high, the first-price auction generates higher revenue than the SPA.

Figure 4 shows a numerical example of the expected revenues when $c_i \sim U[1, 2]$, $i = 1,2$ and $s \sim U[-z, z]$. When the recovery rate r is zero (shown as solid and dashed lines) so that bankruptcy is very costly, the FPA always yields the higher revenue regardless of the risk parameter z. In the case of $r = 1$ (shown as dotted lines) so that bankruptcy is less costly, the procurer becomes risk-loving and the SPA becomes more attractive when the risk is small (approximately when $z < 0.55$).

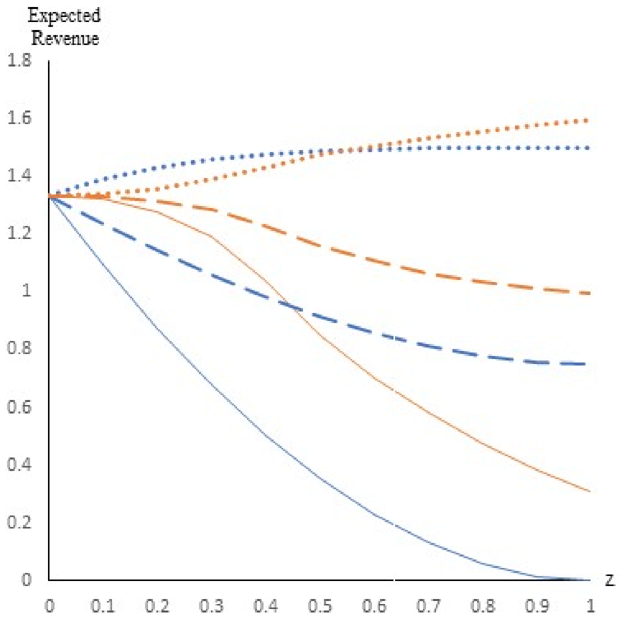
Table 1 summarizes the revenue result.

TABLE 1. Bankruptcy Cost, Risk and Revenue

This table summarizes the main revenue results of this paper under different bankruptcy costs and size of the risk. The third column compares expected revenue of the procurer under the first-price auction (FPA) and under the second-price auction (SPA)

Bankruptcy Cost	Risk	FPA vs. SPA
High $i.e.$ (d, r) close to $(0,0)$	Large $i.e.$ large z	FPA
Low $i.e.$ (d, r) close to $(1,1)$	Small $i.e.$ small z	SPA

r is the recovery rate or the portion of the payment to the winner that is reclaimed by the procurer in the presence of bankruptcy. d is the completion rate or the portion of production completed at the time of the bankruptcy declaration.



	Recovery %	Completion %	Bankruptcy Cost
Dotted	$r=1$	$d=0.5$	Low
Dashed	$r=0$	$d=0.5$	High
Solid	$r=0$	$d=0$	Highest

FIGURE 4. Expected Revenue in the FPA and SPA with Different Bankruptcy Risk and Cost

Procurer's expected revenues for the FPA(orange) and the SPA(blue) when $v_0=3$ and $c_i \sim U[1,2]$, $i=1,2$ in each recovery rate r and completion rate d.

6. CONCLUSION

This paper investigates a special case of two-bidder procurement with asymmetric post-auction bankruptcy risk, yielding an asymmetric equilibrium. I have shown that in the presence of asymmetric post-auction bankruptcy risk, procurer's value - relative to her expected payment - upon the good or project is another factor determining her revenue maximizing mechanism. As long as the full completion of the project is guaranteed ($i.e.$ the completion rate is less than 1), there is a threshold value of her WTP at which exceeding it will make the FPA a more lucrative choice. Procurers should meticulously evaluate the project's value compared to the expected cost because as opposed to the auctioneers in high-bid auctions, procurers start the auction without the item in their possession. Despite the asymmetric bidders involved, it is consistent with the past literature - where symmetric bidders were assumed - that the FPA is more favorable if the cost of bankruptcy and the magnitude of risk are large. However, this is only true when the procurer's WTP is above the threshold.

Moreover, during the analysis of the asymmetric BNE bidding strategies, I confirm that as opposed to the past literature, the FPA induces a cheaper procurement price when bankruptcy risk is large. Moreover, the bidder exposed to the post-auction shock - if his private valuation is quite advantageous - is quite confident of winning that he may bid less aggressive than his counterpart and may not even take the risk of default at all. This is in contrast to our intuition of the past literature that the advent of the option to declare bankruptcy renders the bidder more aggressive. The role of asymmetry is distinct although my analysis can only reach a special realm. Generalization of this model is left for further research.

APPENDIX

A. PROOF OF PROPOSITION 1

PROPOSITION 1: *Consider a two-bidder SPA. Assume independently distributed* $c_i \sim F, f$ *over* $[\underline{c}, \bar{c}]$ *and* $s \sim G, g$ *over* $[-z, z]$ *and* $z < \underline{c}$. *Then, the BNE bidding strategies of bidder 1 and 2 are* $B_1(c_1) = c_1$ *and* $B_2(c_2) = c_2 - z \quad \forall c_1, c_2 \in [\underline{c}, \bar{c}]$

PROOF: As shown in Board(2007), $B_2(c_2)$ is given by $E(\max\{B_2(c_2) - c_2 - s, 0\}) \equiv 0 \quad \forall c_2 \in [\underline{c}, \bar{c}]$.

$$\Rightarrow \int_{-z}^{z} \max\{B_2(c_2) - c_2 - s, 0\} g(s) ds \equiv 0$$

$$\Rightarrow \int_{-z}^{m} (B_2(c_2) - c_2 - s) g(s) ds \equiv 0 \text{ for } B_2(c_2) - c_2 \geq -z \text{ (otherwise zero)}$$

$$\Rightarrow (B_2(c_2) - c_2) \int_{-z}^{m} g(s) ds - \int_{-z}^{m} s g(s) ds \equiv 0 \quad \text{where } m = \min\{B_2(c_2) - c_2, z\}.$$

$$\Rightarrow (B_2(c_2) - c_2) G(m) - \left([sG(s)]_{-z}^{m} - \int_{-z}^{m} G(s) ds \right) \equiv 0$$

$$\Rightarrow (B_2(c_2) - c_2 - m) G(m) + \int_{-z}^{m} G(s) ds \equiv 0$$

If $m = z$, then $(B_2(c_2) - c_2 - m) G(m) + \int_{-z}^{m} G(s) ds = (B_2(c_2) - c_2 - z) + \int_{-z}^{z} G(s) ds > 0$ which is a contradiction. If $m = B_2(c_2) - c_2$, then $(B_2(c_2) - c_2 - m) G(m) + \int_{-z}^{m} G(s) ds = \int_{-z}^{B_2(c_2) - c_2} G(s) ds$ which can only be zero when $B_2(c_2) = c_2 - z$.

□

B. PROOF OF THE BOUNDARY CONDITION IN THE FPA

Suppose $b_1(c_1)$ and $b_2(c_2)$ are the BNE bidding strategies of bidder 1 and 2 in the FPA. Then, the boundary condition is $\bar{c} - z < b_2(\bar{c}) < b_1(\bar{c}) \leq \bar{c}$.

PROOF: Let $u_i(c_i) \equiv \max_{b_i} u_i(b_i; c_i)$ for $i = 1,2$. Suppose both bidders have their highest possible valuation, *i.e.* $c_1 = c_2 = \bar{c}$. Then, it is clear that bidder 1's expected utility is zero, that is, $u_1(\bar{c}) = (b_1(\bar{c}) - \bar{c})[1 - F(\phi_2(b_1(\bar{c})))] = 0$ which means either ① $b_1(\bar{c}) = \bar{c}$ or ② $1 - F(\phi_2(b_1(\bar{c}))) = 0$. ② implies $b_1(\bar{c}) \geq b_2(\bar{c})$.

There is a possibility of a favorable shock and therefore bidder 2's expected utility is positive when $c_2 = \bar{c}$, *i.e.* $u_2(\bar{c}) > 0$,

$$i.e. \quad u_2(b_2(\bar{c}); \bar{c}) = \left(\int_{-z}^{b_2(\bar{c}) - \bar{c}} G(s) ds \right) [1 - F(\phi_1(b_2(\bar{c})))] > 0$$

which means ③ $b_2(\bar{c}) - \bar{c} > -z$ *and* ④ $1 - F(\phi_1(b_2(\bar{c}))) > 0$. ④ implies $b_2(\bar{c}) < b_1(\bar{c})$. Combining ① to ④, we can conclude that $\bar{c} - z < b_2(\bar{c}) < b_1(\bar{c}) \leq \bar{c}$.

□

C. PROOF OF PROPOSITION 2

PROPOSITION 2: *Assume* $c_i \sim F, f$ *over* $\left[\underline{c}, \bar{c}\right]$ *and* $s \sim G, g$ *over* $[-z, z]$ *with* $E(s) = 0$. *Further assume* z *is large enough so that* $I_B = [\underline{c}, \bar{c}]$ *holds. Then, (1)* $\phi_1(b)$ *and* $\phi_2(b)$ *intersect at most once. (2) And if they do not intersect,* $b_1(c) > b_2(c)$ *for all* $c \in \left[\underline{c}, \bar{c}\right]$

PROOF: If $\exists b_0 \in [\underline{b}, \bar{b})$ where $\underline{b} \equiv \max\{b_1(\underline{c}), b_2(\underline{c})\}$ and $\bar{b} \equiv \min\{b_1(\bar{c}), b_2(\bar{c})\}$ such that $\phi_1(b_0) = \phi_2(b_0) \equiv \hat{c}$, then,

$$\phi_2{}'(b_0) = \frac{1-F(\hat{c})}{f(\hat{c})} \frac{1}{(b_0 - \hat{c})} > \frac{1-F(\hat{c})}{f(\hat{c})} \frac{G(b_0 - \hat{c})}{\left(\int_{-z}^{b_0 - \hat{c}} G(s) ds\right)} = \phi_1{}'(b_0)$$

which means

$$\frac{\left(\int_{-z}^{b_0 - \hat{c}} G(s) ds\right)}{G(b_0 - \hat{c})} > b_0 - \hat{c}.$$

This holds because we can show that

$$\frac{\int_{-z}^{b_0 - \hat{c}} G(s) ds}{G(b_0 - \hat{c})} > \int_{-z}^{b_0 - \hat{c}} G(s) ds > b_0 - \hat{c}.$$

The first inequality is due to $b_0 < \hat{c} + z$ and $0 < G(b_0 - \hat{c}) < 1$. The second one is due to $E(s) = 0$. Since $\phi_i{}'(b_0) = 1/b_i{}'(\hat{c})$, it implies if there exists $\hat{c} \in [\underline{c}, \bar{c})$ such that $b_1(\hat{c}) = b_2(\hat{c})$ then $b_1{}'(\hat{c}) > b_2{}'(\hat{c})$. In other words, if the two bid curves ever intersect, the former is steeper, which means they intersect at most once. By boundary condition that $b_2(\bar{c}) < b_1(\bar{c})$, it implies that if they do not intersect, then $b_1(c) > b_2(c)$ for all $c \in \left[\underline{c}, \bar{c}\right]$ and thus, $\phi_1(c) < \phi_2(c)$.

□

D. PROOF OF COROLLARY 1

COROLLARY: *Assume* $c_i \sim F, f$ *over* $\left[\underline{c}, \bar{c}\right]$ *and* $s \sim U[-z, z]$ *with* $0 < z \le \underline{c}$ *and* $z \le \bar{c} - \underline{c}$. *Further assume* z *is large enough so that so that* $I_B = [\underline{c}, \bar{c}]$ *holds. Then,* $\phi_1(b)$ *and* $\phi_2(b)$ *intersect at most once. If they do not intersect,* $b_1(c) > b_2(c)$ *for all* $c \in \left[\underline{c}, \bar{c}\right]$ *and in addition, the distribution of bidder 2's bids dominate that of bidder 1's in terms of the hazard rate.*

PROOF: If $\exists b_0 \in [\underline{b}, \bar{b})$ where $\underline{b} \equiv \max\{b_1(\underline{c}), b_2(\underline{c})\}$ and $\bar{b} \equiv \min\{b_1(\bar{c}), b_2(\bar{c})\}$ such that $\phi_1(b_0) = \phi_2(b_0) \equiv \hat{c}$, then

$$\phi_2{}'(b_0) = \frac{1-F(\hat{c})}{f(\hat{c})} \frac{1}{(b_0 - \hat{c})} > \frac{2\left(1-F(\hat{c})\right)}{f(\hat{c})} \frac{1}{\left(b_0 - \hat{c} + z\right)} = \phi_1{}'(b_0).$$

This is clear due to $b_0 < \hat{c} + z$ and first order conditions.

Since $\phi_i{}'(b_0) = 1/b_i{}'(\hat{c})$, it implies if there exists $\hat{c} \in [\underline{c}, \bar{c})$ such that $b_1(\hat{c}) = b_2(\hat{c})$ then $b_1{}'(\hat{c}) > b_2{}'(\hat{c})$. In other words, if the two bid curves ever intersect, the former is steeper, which means they intersect at most once. By the boundary condition that $b_2(\bar{c}) < b_1(\bar{c})$, it implies that if they do not intersect, then $b_1(c) > b_2(c)$ for all $c \in \left[\underline{c}, \bar{c}\right]$ and thus, $\phi_1(c) < \phi_2(c)$ in which case, for all $b \in (\underline{b}, \bar{b})$,

$$b-\phi_1(b)=\frac{1-H_2(b)}{h_2(b)}>\frac{2(1-H_1(b))}{h_1(b)}-z=b-\phi_2(b).$$

□

E. PROOF OF PROPOSITION 3

PROPOSITION 3: *Consider a FPA. Assume* $c_i\sim U[\underline{c},\bar{c}]$ *and* $s\sim U[-z,z]$. *Further assume that* z *is large enough so that* $I_B=[\underline{c},\bar{c}]$ *holds. Then,*

$$b_1(c_1)=\frac{c_1+\bar{c}}{2}-\frac{z}{4}\ \text{ and }\ b_2(c_2)=\frac{c_2+2\bar{c}}{3}-\frac{z}{2}\ \text{ for }\ \forall z\in\left[\frac{4(\bar{c}-\underline{c})}{9},\bar{c}-\underline{c}\right]$$

PROOF: Suppose z is large enough so that bidder 2 cannot bid to a point where bankruptcy risk is zero. Then, bidder 2's expected utilities can be written as

$$u_2(b_2;c_2)=\left(\int_{-z}^{b_2-c_2}G(s)ds\right)\left[1-F(\phi_1(b_2))\right].$$

Then, the first order conditions become

$$1-F(\phi_2(b_1))-(b_1-c_1)f(\phi_2(b_1))\phi_2{}'(b_1)=0\ \text{ at }\ b_1=b_1(c_1)$$

and

$$G(b_2-c_2)(1-F(\phi_1(b_2)))=\left(\int_{-z}^{b_2-c_2}G(s)ds\right)f(\phi_1(b_2))\phi_1{}'(b_2)\ \text{ at }\ b_2=b_2(c_2).$$

Using uniform distribution assumptions, the FOC can be summarized as

$$b_1=c_1+\frac{\bar{c}-\phi_2(b_1)}{\phi_2{}'(b_1)}\ \text{ and }\ b_2=c_2-z+\frac{2(\bar{c}-\phi_1(b_2))}{\phi_1{}'(b_2)}.$$

It can be shown that $b_1(c_1)=\frac{c_1+\bar{c}}{2}-\frac{z}{4}$ and $b_2(c_2)=\frac{c_2+2\bar{c}}{3}-\frac{z}{2}$ or $\phi_1(b)=2b-\bar{c}+z/2$ and $\phi_2(b)=3b-2\bar{c}+(3z)/2$ solve the equations. In this case, the large z satisfies $\frac{4(\bar{c}-\underline{c})}{9}\le z$.

□

F. PROOF OF PROPOSITION 4

PROPOSITION 4: *Consider a FPA with* $c_i\sim U[\underline{c},\bar{c}]$ *and* $s\sim U[-z,z]$. *Assume that* z *is small enough so that* $I_B=[\underline{c},\bar{c}]$ *does not hold. Then, for* $z<4(\bar{c}-\underline{c})/9$,

$$b_1(c_1)=\frac{(c_1+\bar{c})}{2}-\frac{\hat{a}_2}{2}z\ \text{ and }\ b_2(c_2)=\begin{cases}\dfrac{(c_2+\bar{c})}{2}-\dfrac{\hat{a}_2}{4}z & \text{if}\quad c_2<c_0\\[2ex] \dfrac{(c_2+2\bar{c})}{3}-\dfrac{(1+\hat{a}_2)}{3}z & \text{if}\quad c_2\ge c_0\end{cases}$$

where $c_0=\bar{c}-\frac{z}{2}\hat{a}_2-2z$ *and* $\hat{a}_2=\left[(3(\underline{c}-\bar{c})-z)+\sqrt{(3(\bar{c}-\underline{c})+z)^2+8z^2}\right]/z$.

In particular, (1) $b_2(c_0)=c_0+z$, *(2) if* $z\to 4(\bar{c}-\underline{c})/9$, *then* $\hat{a}_2\to 1/2$ *with which the BNE strategies are equivalent to those in Proposition 3, and (3) if* $z\to 0$, *then* $\hat{a}_2\to 0$ *with which the BNE strategies are equivalent to those in the FPA with no bankruptcy risk:* $b(c_i)=(c_i+\bar{c})/2\quad\forall c_i$.

PROOF: Suppose z is small enough so that bidder 2 can bid to a point where bankruptcy risk is zero. Let c_0 be such that $b_2(c_0)=c_0+z$ and denote $b_0 \equiv b_2(c_0)$. Let $\phi_1(b)=2b-a_1\bar{c}+a_2 z$, then bidder 2's response is

$$\phi_2(b)=\begin{cases} 2b-\dfrac{(1+a_1)}{2}\bar{c}+\dfrac{a_2}{2}z & \text{if} \quad b<b_0 \\ 3b-(1+a_1)\bar{c}+(1+a_2)z & \text{if} \quad b\geq b_0 \end{cases}.$$

Let $p\equiv \Pr(c\leq c_0)$: probability of bidder 2's zero bankruptcy risk, *i.e.* p=*probability of* $b_2(c_2)\geq c_2+z$. Then, $p=F(c_0)=\dfrac{c_0-\underline{c}}{\bar{c}-\underline{c}}$ if $c_0\in[\underline{c},\bar{c}]$ (1 or 0 otherwise). And bidder 1's expectation on bidder 2's valuation is

$$\begin{aligned} E(\phi_2) &= p\left(2b-\frac{(1+a_1)}{2}\bar{c}+\frac{a_2}{2}z\right)+(1-p)\left(3b-(1+a_1)\bar{c}+(1+a_2)z\right) \\ &= [2p+3(1-p)]b-\left[\frac{(1+a_1)p}{2}+(1+a_1)(1-p)\right]\bar{c}+\left[\frac{a_2}{2}p+(1+a_2)(1-p)\right] \end{aligned}$$

Then, bidder 1's response is

$$\phi_1(b)=b+\frac{E(\phi_2(b))-\bar{c}}{E(\phi_2{}'(b))}=2b-\frac{\left[\dfrac{(1+a_1)p}{2}+(1+a_1)(1-p)+1\right]}{2p+3(1-p)}\bar{c}+\frac{\left[\dfrac{a_2}{2}p+(1+a_2)(1-p)\right]}{2p+3(1-p)}z$$

For ϕ_1 and ϕ_2 to be the BNE, the second coefficient should equal a_1 and also, the third coefficient should equal a_2. That is

$$\frac{\left[\dfrac{(1+a_1)p}{2}+(1+a_1)(1-p)+1\right]}{2p+3(1-p)}=a_1 \quad \text{and} \quad \frac{\left[\dfrac{a_2}{2}p+(1+a_2)(1-p)\right]}{2p+3(1-p)}=a_2$$

$$\Rightarrow \quad ① \ a_1=1 \ \text{and} \ ② \ p=\frac{2(2a_2-1)}{a_2-2}$$

Since $b_0\equiv b_2(c_0)=c_0+z$, then $\phi_2(b_0)+z=b_0$ holds. That is, $3b_0-(1+a_1)\bar{c}+(1+a_2)z+z=b_0$.

Using ①, we get $b_0=\bar{c}-\dfrac{z}{2}a_2-z$ and thus, $c_0=\bar{c}-\dfrac{z}{2}a_2-2z$. Since $p=\dfrac{c_0-\underline{c}}{\bar{c}-\underline{c}}$, we have

$$③ \ p=\max\left\{0,\min\left\{1,\frac{\bar{c}-\dfrac{z}{2}a_2-2z-\underline{c}}{\bar{c}-\underline{c}}\right\}\right\}$$

By ② and ③,

$$p=\frac{2(2a_2-1)}{a_2-2}=\frac{\bar{c}-\dfrac{z}{2}a_2-2z-\underline{c}}{\bar{c}-\underline{c}}.$$

Then, using quadratic formula, we can solve for a_2 as proposed. Now, we have all parameters in the BNE strategies.

In this case, the small enough z satisfies $z<4(\bar{c}-\underline{c})/9$. And it can be shown that as $z\rightarrow 4(\bar{c}-\underline{c})/9$, $\hat{a}_2$ converges to $1/2$ and p converges to zero with which the BNE strategies are those in Proposition 3. Furthermore, it can be shown that as $z\rightarrow 0$, $\hat{a}_2$ converges to zero and

p converges to 1 with which the BNE strategies are those in FPA without bankruptcy risk. □

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